Momentum in Special Relativity

Study Material By: Sunil Kumar Yadav*

Department of Physics,

Maharaja College, Ara,

Bihar 802301, India.

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"No one must think that Newton's great creation can be overthrown in any real sense by this [Theory of Relativity] or by any other theory. His clear and wide ideas will for ever retain their significance as the foundation on which our modern conceptions of physics have been built."

— Albert Einstein

In Newtonian mechanics, we have seen that momentum conservation law remains invariant under Galilean transformation. The Newtonian definition of momentum is given by $\mathbf{p} = m\mathbf{u}$, where m is mass of object and $\mathbf{u}$ is velocity. One may ask what happens to the momentum conservation law if we use Lorentz transformation equations? The old definition of linear momentum is valid or we need to redefine this. To see this, below we consider an example of elastic collision of two objects.

In Fig. (1), we have demonstrated an elastic collision of two particles A and B. The S frame is considered to be at rest and S' frame moving in positive x-direction with speed u. Observers of both the frames records the elastic collision between the objects A and B. We assume that before collision, object A is at rest in S frame and object B is at rest in S' frame. Next, we consider that at the same moment, particle A has been thrown to positive y-direction and particle B to negative y'-direction with speed U_A and U'_B respectively. We

*Electronic address: sunil.phy30@gmail.com

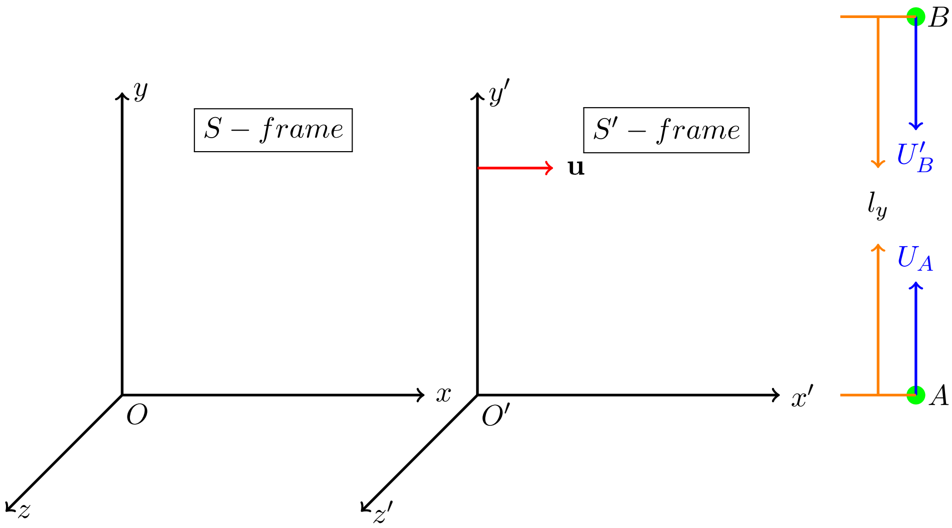

FIG. 1: S' frame moving with constant velocity $\mathbf{u}$ relative to S frame in the +ive $x-$direction, where $\mathbf{u} = (u, 0, 0)$. On the right-hand side we have shown an elastic collision of two objects A and B along y direction. Initially, for both frames, the objects are l_y distance apart.

take that

$$U_A = U'_B, \tag{1}$$

which implies that particle B behaves in same manner in S' frame as particle A in S frame. After collision, object A bounce back in negative y-direction at speed U_A while object B bounce back in positive y'-direction at speed U_B. If we assume that objects are pushed from positions l_y apart, then for S frame observer collision occurs at $y = l_y/2$ and for S' frame observer it happens at $y' = y = l_y/2$. Thus, measured round trip time τ_0 for S frame observer is given by

$$\tau_0 = l_y/U_A, \tag{2}$$

and for S' frame observer measured time is same and given by

$$\tau_0 = l_y/U'_B. \tag{3}$$

If we consider that round trip time for object B measured in S frame is τ, then speed U_B in S frame is given by

$$U_B = l_y/\tau. \tag{4}$$

The round trip time for particle B in frame S' is τ_0 as we have already discussed. The relation between τ and τ_0 is given by

$$\tau = \gamma\tau_0, \tag{5}$$

where $\gamma = 1/\sqrt{1-\beta^2}$ with $\beta = u/c$. Now from Eq. (2)

$$U_A = l_y/\tau_0, \tag{6}$$

and using Eq. (5) in Eq. (4), we obtain

$$U_B = \frac{l_y\sqrt{1-\beta^2}}{\tau_0}. \tag{7}$$

Next, using Newtonian definition of momentum, momentum observed in S frame for objects A and B are respectively given by

$$p_A = m_A U_A = m_A(l_y/\tau_0) \tag{8}$$

and

$$p_B = m_B U_B = m_B\sqrt{1-\beta^2}(l_y/\tau_0), \tag{9}$$

where m_A and m_B are the measured masses in S frame. From Eqs. (8) and (9), we see that momentum conservation will be violated if $m_A = m_B$. In order to have momentum conservation, we must have

$$m_B = \frac{m_A}{\sqrt{1-\beta^2}}. \tag{10}$$

Next, if we consider the case that velocities U_A and U_B tends to zero, then we can obtain how the relativistic mass of an object varies with velocity. If U_A and U_B are negligible compared to u, i.e., $U_A, U_B \ll u$, then S frame observer will witness that object B approach to A with speed u and makes continuous glancing collision. Now for $U_A = 0$, if we assume that m_0 is the mass of object A in S frame, then $m_0 = m_A$. Here m_0 is the Newtonian mass and now we call it rest mass. Again for $U'_B = 0$, if we take m is the mass of object A in S frame, then $m = m_B$. Thus, we can write Eq. (10) as,

$$m = \frac{m_0}{\sqrt{1-u^2/c^2}} = \frac{m_0}{\sqrt{1-\beta^2}} = \gamma m_0 \tag{11}$$

Now the relativistic momentum is defined by

$$p_{rel} = \frac{m_0 u}{\sqrt{1-u^2/c^2}} = \frac{m_0 u}{\sqrt{1-\beta^2}} = \gamma m_0 u. \tag{12}$$

Similarly, one can redefine the Newton's law of motion, relativistic acceleration, etc. In next article, we study the mass energy relation.

[1] R. Resnick, *Introduction to Special Relativity*, Wiley-VCH, (1968) .

[2] A. Beiser, *Concepts of modern physics*, Tata McGraw-Hill Education, (2003).